Sorting Continuous-Time Signals: Analog Median and Median-Type Filters

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Abstract—There are two natural orderings in signals: temporal order and rank order. There is no compelling reason to explore only one of these orderings, either in the discrete-time or in the continuous-time case. Nevertheless, the concept of rank order for continuous-time signals remains virtually unstudied, which is in striking contrast to the discrete-time case: ranked order discrete-time filters, of which the running median is the most common example, have been intensively studied for three decades. The dependence of these nonlinear systems on the order statistics of the input samples stands in contrast with the tapped delay line filter, which depends on temporal order only. However, continuous-time signals can also be meaningfully sorted: a fact that is explored in this paper to define and study the analog median filter and other ranked-order filters. The paper introduces the basic tools needed to analyze and understand these continuous-time nonlinear filters (the distribution function and the sorting) and presents some of their properties in a tutorial way. The analog median filter is defined in terms of the (unique) nonincreasing left-continuous sorting. More general filters can also be defined, including filters similar to α -trimmed mean filters and L filters. These include filters that depend on one parameter and contain the running average and running median as special cases. The rate of convergence of the digital median filter to the analog median filter is discussed and related to the signal sampling period, the duration of the filter window, and the smoothness of the input signal. The paper introduces the concept of noise width and studies the effect of additive and multiplicative noise at the output of the analog median filter in terms of the noise width and the smoothness of the input signal.

Index Terms—Analog circuits, analog systems, approximation methods, continuous-time filters, continuous-time systems, discrete-time filters, discrete-time systems, error analysis, median filters, nonlinear filters, sorting.

I. INTRODUCTION

A. Motivation

TEMPORAL order and rank order are the two natural orderings that emerge when dealing with signals. For discrete-time signals, both orderings have been thoroughly exploited, leading to fundamental research on linear and nonlinear digital filters. We see no compelling reason to neglect one of the two orderings, regardless of the nature of the signals (discrete-time or continuous-time). However, as far as we know, the concept of rank order for analog, continuous-time signals, which naturally leads to analog median and median-related filters, remains virtually unexplored.

The only works on the analog median filter of which we are aware are [1] and [2]. Neither paper generalizes the concept of

sorting to continuous-time signals. Longbotham and Bovik [2] show by means of a counterexample that some properties of the filters defined in [1] are more subtle than they initially seem. The issues raised in [2] that are related to the nature and existence of root signals of the median filter as defined in [1] were not addressed until recently [3]. The lack of theoretical results on the analog median filter was noted, for example, in [4].

The almost complete inexistence of a theoretical background for the analog median filters when compared with the maturity of the theory for the corresponding digital filters is in striking contrast to the state-of-the-art in neural networks, which is another class of nonlinear systems in which both continuous-time and discrete-time models are well developed. This state of affairs is perhaps due to the additional difficulties related to the definition of a sorting operation for continuous-time signals when compared with the discrete-time case.

Fitch *et al.* [1] mention several reasons for extending the theory of median filtering from the digital to the analog domain, including the need for fast implementations. Several analog implementations of the median filter have been reported, but they should not be confused with true single-input single-output analog median filters, and shed no light on the meaning of rank orderings for continuous-time signals. However, the analog implementations allow simpler and faster circuits, without A/D or D/A converters, yielding large power and circuit area savings. This is particularly attractive in applications such as mobile telecommunications, in which power and circuit area are severely limited resources. These advantages might be shared by true analog median filters. The need for a device that would perform two-dimensional (2-D) median filtering, included as a stage in the optical section at the front end of the imaging system, is mentioned in [1]. A hybrid optical/electronic system has in fact been described [5], but it is still a step away from the analog, optical median filter. Although the implementation issues are not the main purpose of this paper, we will discuss them briefly in Section VI in the hope of stimulating further research in that important direction.

We believe in the benefits of bringing the theory of ranked-order filters to a more symmetrical state in what concerns the discrete-time and the continuous-time cases. Nonlinear systems are notoriously difficult to understand, and a study of the continuous-time rank order and median filter brings to scene new analytical tools that may lead to a deeper understanding of an important class of nonlinear systems.

B. Purpose of This Work

We intend to explore the concept of rank ordering for continuous-time signals, bridging the gap between analog and digital

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median and related filters. It is possible to develop a consistent and rigorous continuous-time rank order theory, which contains the existing discrete-time theory as a special case.

We choose to formulate the analog median and median-type filters in terms of the concepts of distribution and sorting. This brings out new insights and a more general framework within which to study median-type filters. The nonequivalent definitions of the analog median filter in [1], and the questions raised in [2] concerning the nature of its root signals, are examples of issues that can be understood in terms of the framework described in this paper. They are related to the left or right continuity of the sorting of the windowed signal at one half the window width [3].

The relation between the discrete-time and continuous-time median filter are another example of the usefulness of the new perspective. This was first discussed in [1], but the new tools used in this paper allow us to discuss the convergence rate of the digital to the analog filter in terms of the smoothness of the input signal.

The concepts of distribution function and sorting, upon which our formulation of the analog median filter rests, appear to be new to the engineering literature. The sorting or nonincreasing rearrangement was introduced by Hardy and Littlewood, and some results concerning it can be found in their classical work on inequalities written with Pólya [6] or scattered among mathematical books as diverse as Zygmund's treatise on trigonometric series [7], Lorentz's book on the Bernstein polynomials [8] or Stein and Weiss's book [9] on Fourier analysis in Euclidean spaces. Be aware that the conventions and notations used differ from author to author (some of the possibilities related to right continuity or left continuity are noted below) and that the material in those works is often condensed down to the essential.

To make the paper reasonably self-contained, we include an introduction to the concept of distribution function and sorting in Sections II and III. The properties needed are established in the Appendixes. We are not aware of any single paper or book that discusses all the results that we needed and derived. Some may be new.

We deliberately avoided the strict "lemma/theorem/corollary" type of presentation. We tried to emphasize ideas rather than details; if we were successful, the chain of thought that guided us to the main results should be apparent to the reader.

Interpreting the analog median filter in terms of the sorting of a continuous-variable signal leads to a broader view of the area and to other ranked-order filters and clarifies the connection between the analog and digital cases, the role of the continuity of the sorting and its impact on the root signals, and the behavior under noise. The present paper is a contribution toward this program: to understand the continuous-time approach and the associated analytical tools and to explore them in order to obtain new insights about these nonlinear systems.

C. Mathematical Background and Notation

Some familiarity with the basics of measure theory is assumed, but the intuitive concept of "measure" should be sufficient to follow most of the paper. The readers unacquainted with measure theory should note that the Lebesgue measure of an interval $[a, b]$ is $b - a$ and that the measure of a union of

disjoint intervals is the sum of the measures of each interval. If a property is valid except on a set of measure zero, we say (as usual) that it is valid "almost everywhere." All functions are tacitly assumed to be measurable.

The characteristic function of $S \subset \mathbb{R}$ is denoted by χ_S . Hence

$$\chi_S(x) := \begin{cases} 1, & x \in S \\ 0, & x \notin S. \end{cases}$$

The modulus of continuity $\omega_f(\delta)$ of a continuous function f is defined by

$$\omega_f(\delta) := \sup_{|a-b| \leq \delta} |f(a) - f(b)|$$

and converges to zero when $\delta \rightarrow 0$. The condition $\omega_f(\delta) \leq \alpha\delta^\gamma$, where α and γ are constants, leads to the class of functions f that satisfy a Lipschitz condition of order γ and constant α :

$$|f(a) - f(b)| \leq \alpha|a - b|^\gamma.$$

This condition is useful only if $0 < \gamma \leq 1$. For details about these concepts, see any standard book on approximation theory.

II. DISTRIBUTION FUNCTION

A. Definition and Basics

Given a function $f: \mathbb{R} \rightarrow \mathbb{R}$, assume that somehow, we were able to determine a new function denoted by $\bar{f}$, which corresponds to the idea of " f sorted by nonincreasing order." It is intuitively obvious that both f and $\bar{f}$ should have the same "distribution of amplitudes." To render this concept precise, consider the distribution function M_f associated with f , which is defined by

$$M_f(y) := \text{meas}\{x: f(x) > y\}$$

where "meas S " denotes the Lebesgue measure of the set S . It is tacitly assumed that x belongs to the domain of f and that M_f is finite almost everywhere. The latter hypothesis avoids certain theoretical difficulties, which cannot occur in the context of analog median type filters anyway.

The distribution cannot usually be expressed in closed form. Consider for example $f(x) = axe^{-bx}$. Its distribution function can be numerically computed (see Fig. 1), but there appears to be no general simple closed-form expression for the roots of $xe^{-x} = y$. An even more complex case is illustrated in Fig. 2.

There are two remarks that we wish to make immediately. The first is about the irrelevance of the behavior of f at its points of discontinuity (in fact, arbitrarily modifying f in a set of zero measure does not change M_f). The second concerns the shift-invariance of the map $f \mapsto M_f$, which is a consequence of the shift-invariance of the measure. Thus, the functions $f(x)$ and $g(x) = f(x - \tau)$ have the same distribution.

However, two functions f and g that differ on sets of positive measure may have the same distribution, even if $f(x) \neq g(x - \tau)$. For example, let S and S' be two sets of measure $a > 0$. Then, χ_S and $\chi_{S'}$ have the same distribution function

$$M_{\chi_S}(y) = M_{\chi_{S'}}(y) = \chi_I(y) \quad (1)$$

Fig. 1. Left: Two simple functions f and g . Right: Their distributions M_f and M_g .

Fig. 2. More complex function (solid line) and its distribution (dashed line).

where $I = (0, a)$. Two or more functions with the same distribution are called “equimeasurable” and can be considered to be “rearrangements” of each other. A function $f(x)$ and its “reversed” version $f(-x)$ are equimeasurable. A more general class of equimeasurable functions can be obtained from a single function f as follows: Split f in a number of components f_i with disjoint supports¹ such that

$$f = \sum_i f_i$$

and then translate, reverse, or permute the f_i at will, without overlapping any two of them. The function obtained will be equimeasurable with f .

To close this section, we note that there are other reasonable definitions of the distribution function. For example, the definition $M_f(y) := \text{meas}\{x: f(x) \geq y\}$, which leads to a left-continuous distribution function. There are no essential distinctions between these definitions, in the sense that any proofs derived in one case can be readily adapted to the other.

We are now ready to discuss the meaning of “sorting” for continuous-time signals.

¹The support of a function f is the closure of the set where $f \neq 0$. Roughly speaking, signals with disjoint supports do not overlap.

Fig. 3. Sorting of a function is unique. Two distinct sortings $\bar{g}$ and $\bar{f}$ cannot be equimeasurable. If $\bar{g}(x_0) > \bar{f}(x_0)$, the left-continuity implies the existence of an interval $[x_1, x_0]$ in which $\bar{f} < \bar{g}(x_0) - \epsilon$, and this can be used to show that $\bar{g}$ and $\bar{f}$ are not equimeasurable, which is a contradiction.

III. SORTING CONTINUOUS-TIME SIGNALS

Now that we have defined the distribution function M_f of a given f , we look for nonincreasing (or nondecreasing) and left-continuous (or right-continuous) functions that are equimeasurable with f . To fix ideas, we consider the nonincreasing left-continuous case only. As a first step, we show that the solution is unique; we will denote it by $\bar{f}$, and call it the sorting of f . The second and final step is constructive: We give a formula for $\bar{f}$.

A. Uniqueness

Given f , we maintain that its nonincreasing left-continuous sorting $\bar{f}$ is unique. Assume to the contrary that there exist two different equimeasurable sortings $\bar{f}$ and $\bar{g}$ of the given f . Without loss of generality, let $\bar{g}(x_0) > \bar{f}(x_0)$ for some x_0 (see Fig. 3). Pick $\epsilon > 0$ such that $\bar{g}(x_0) - \epsilon > \bar{f}(x_0)$. By the left continuity of $\bar{f}(x)$, there exists an interval $[x_1, x_0]$ in which $\bar{f}(x) < \bar{g}(x_0) - \epsilon$. Then

$$\text{meas}\{x: \bar{g}(x) > \bar{g}(x_0) - \epsilon\} \geq x_0$$

whereas

$$\text{meas}\{x: \bar{f}(x) > \bar{g}(x_0) - \epsilon\} \leq x_1 < x_0.$$

The quantities on the left side are the values at the point $p = \bar{g}(x_0) - \epsilon$ of the distribution functions of $\bar{g}$ and $\bar{f}$, respectively. The inequalities, therefore, show that $\bar{f}$ and $\bar{g}$ are not equimeasurable, which is a contradiction. Consequently, the (left-continuous nonincreasing) sorting is unique.

The proof can be readily modified to show that the nonincreasing right-continuous sorting is also unique. The two sortings will be equal, except at their points of discontinuity (they will be equal everywhere provided they are continuous). There is an essentially unique way of sorting a given function, and all the ambiguity can be removed by imposing, for example, a left- or right-continuity constraint.

B. Sorting as the Inverse of M_f

We now proceed to the second step: the construction of $\bar{f}$. It turns out that $\bar{f}$ is, essentially, the inverse function of its distribution function M_f . This can be seen as follows: Recall that splitting the domain of a given f in a finite number of nonoverlapping intervals, and permuting, translating, or time-reversing the “pieces” of f without overlapping them, leads to a new function, which has the same distribution function M_f . Because M_f is invariant under these operations and is nonincreasing, one might be tempted to take it as the sorting of f . However, this

Fig. 4. (Right-continuous) distribution function $M_f(y) = \text{meas}\{x: f(x) > y\}$. The sorting $\bar{f}$ of f is the inverse function of M_f properly defined to account for discontinuities: $\bar{f}(x) := \inf_y \{M_f(y) < x\}$.

is wrong. If $f(t)$ is interpreted as, say, “current as a function of time,” then $M_f(y)$ would describe “time as a function of current.” Swapping axes, that is, considering M_f^{-1} instead of M_f , leads to a nonincreasing function with the correct units.

The only difficulty is that M_f , in general, does not have an inverse in the strict sense (it may be constant over certain intervals, as in Fig. 4). Roughly speaking, there are essentially two ways of defining a generalized inverse of M_f : Draw a horizontal line of height x across the plot of M_f , search for the left-most point y still satisfying $M_f(y) < x$, or search for the right-most point y still satisfying $M_f(y) > x$ (again, see Fig. 4).

Both possibilities lead to the usual inverse when M_f is continuous and decreasing. We adopt the first hypothesis, which leads to the definition

$$\bar{f}(x) := \inf_y \{M_f(y) < x\} \quad (2)$$

and the left-continuous sorting. Unless otherwise indicated, from now on, “sorting” means “left-continuous nonincreasing sorting” and, therefore, the function defined by (2). Another reasonable definition is $\hat{f}(x) := \inf_y \{M_f(y) \leq x\}$, which leads to the right-continuous nonincreasing sorting.

To confirm that (2) is in fact the inverse of the distribution function M_f that is properly defined to account for discontinuities, check that $\bar{f}(M_f(y)) = y$ if $M_f(y)$ is a point of continuity of $\bar{f}$ and $M_f(\bar{f}(x)) = x$ if $\bar{f}(x)$ is a point of continuity of M_f . At points of discontinuity, $\bar{f}(M_f(y)) \geq y$, and $M_f(\bar{f}(x)) \leq x$. It can be seen that

$$M_{\bar{f}}(y) := \text{meas}\{x: \bar{f}(x) > y\} = M_f(y)$$

that is, f and its sorting have the same distribution function. They are equimeasurable, as expected, and the integrals of f and $\bar{f}$ are equal

$$\int f(x) dx = \int \bar{f}(x) dx. \quad (3)$$

C. Examples and Remarks

The uniqueness of the sorting immediately shows that a non-increasing f equals its sorting, except possibly at its points of discontinuity. The sorting of a nondecreasing f , on the other hand, is essentially obtained by “reversing” the function $\bar{f}(t) = f(-t)$.

The distribution of the Gaussian $f(x) = e^{-x^2}$ $x \in \mathbb{R}$ is

$$M_f(y) = 2\sqrt{\log \frac{1}{y}}, \quad y \in (0, 1).$$

Fig. 5. One period of $f(x) = \cos x$ and its sorting $\bar{f}$. The area under both curves is the same.

Fig. 6. Two simple examples of distributions and sortings. The behavior of f and g at the discontinuities is not indicated because it does not change the distributions or sortings. Arbitrary translation of f and g or even replacement by the characteristic functions (scaled by A and B) of any set of measure one, would also leave the distributions and sortings invariant. Note that the supports of f and g overlap and that $M_{f+g} \neq M_f + M_g$.

The sorting is $\bar{f}(x) = e^{-x^2/4}$, $x \geq 0$, that is, $\bar{f}(x) = f(x/2)$. This is true for any other even and continuous functions f decreasing away from the origin.

The sorting of sinusoids can also be determined. If $f(x) = \cos x$, $x \in [0, 2\pi]$, then

$$M_f(y) = \text{meas}\{x: \cos x \geq y\} = 2 \arccos y$$

where the function $\arccos x$ is defined to be in $[0, \pi]$. The sorting is

$$\bar{f}(x) = \cos(x/2)$$

as illustrated in Fig. 5. The result also applies for $f(x) = \cos(x + \theta)$, $x \in [0, 2\pi]$ for any real θ .

The sorting of the characteristic function of a set of finite measure is essentially the characteristic function of an interval of the same measure [see (1) and Fig. 6].

The following result will be used in Section IV-B and concerns the sorting of step functions.

Example 1: Consider a piecewise constant left-continuous function defined in $(0, \infty)$, with possible discontinuities at the integers, that is

$$f(x) = c_n, \quad x \in I_n := (n, n+1]$$

Fig. 7. When two functions f and g have disjoint supports $M_{f+g} = M_f + M_g$. The addition of two properly translated copies of the functions f and g of Fig. 6 leads to the function h . The distribution M_h is the sum of distribution functions, and the sorting can be obtained from the inverse (properly defined) of M_h or, equivalently, by sorting. In general, the (nonincreasing) sorting of a step function with values $(A, B, C, \dots)$ can be obtained by sorting the sequence $(A, B, C, \dots)$ by decreasing order.

Fig. 8. (a) Two simple functions f and g and their sortings $\bar{f}$ and $\bar{g}$ (compare with Fig. 1). (b) More complex function (solid line) and its sorting (dashed line; compare with Fig. 2).

where the c_n , $n = 0, 1, 2, \dots$ are real numbers. Using the characteristic functions χ_n of the intervals I_n

$$f(x) = \sum_n c_n \chi_n(x).$$

The sorting of f can be obtained from the sorted sequence $\bar{c}_n$

$$\bar{f}(x) = \sum_n \bar{c}_n \chi_n(x).$$

□

This follows from the uniqueness of $\bar{f}$ and its equimeasurability with f , as illustrated in Fig. 7.

The sortings often cannot be determined in closed form, as in the cases mentioned above. A simple example is $f(x) = ax e^{-bx}$ [the sortings are depicted in Fig. 8(a) and should be compared with the distributions in Fig. 1]. A more complex sorting is illustrated in Fig. 8(b) (compare with Fig. 2).

IV. CONTINUOUS-TIME MEDIAN AND MEDIAN-TYPE FILTERS

We denote by $f_t(x)$ the signal $f(t)$ multiplied by a rectangular window of duration w and centered at t

$$f_t(x) := \begin{cases} f(x), & t - w/2 \leq x \leq t + w/2 \\ 0, & \text{otherwise.} \end{cases}$$

The output g of the median filter is defined by

$$g(t) := \bar{f}_t(w/2).$$

Replacing the evaluation functional at $w/2$ by other functionals of $\bar{f}_t$ leads to other ranked order filters:

$$g(t) := F[\bar{f}_t].$$

In the context of digital filters, this corresponds to the class of L filters [10]. A family of operators depending on one parameter a can be obtained using the functional

$$F_a[h] := \frac{1}{a} \int_{(w-a)/2}^{(w+a)/2} h(x) dx$$

where $h: [0, w] \mapsto \mathbb{R}$, and $0 < a \leq w$. The corresponding filter is defined by

$$g(t) := F_a[\bar{f}_t]. \quad (4)$$

The α -trimmed filters described in [11] are the equivalent digital filters. When $a \rightarrow w$, this filter reduces to the (linear) moving average because of (3) (the integrals of the function and its sorting are equal).

On the other hand, when $a \rightarrow 0$, it is natural to expect that it behaves increasingly more like the median filter. This is clear when the sorting of f_t is continuous at $w/2$, in which case, the absolute value of the difference between the outputs of the median filter and the filter defined by (4) satisfies

$$\begin{aligned} & \left| \frac{1}{a} \int_{(w-a)/2}^{(w+a)/2} \bar{f}_t(x) dx - \bar{f}_t(w/2) \right| \\ & \leq \frac{1}{a} \int_{(w-a)/2}^{(w+a)/2} |\bar{f}_t(x) - \bar{f}_t(w/2)| dx \\ & \leq \omega_{\bar{f}_t}^-(a) \end{aligned}$$

where $\omega_{\bar{f}_t}^-$ denotes the modulus of continuity of $\bar{f}_t$. However, $\omega_{\bar{f}_t}^-(a) \rightarrow 0$ as $a \rightarrow 0$, showing that the output of the filter defined by (4) converges to the output of the analog median filter as $a \rightarrow 0$.

It is shown in Appendix F that if f satisfies a certain Lipschitz condition, so will its sorting [see (16)]. Therefore, if f satisfies a Lipschitz condition of order γ with constant α , then

$$\left| \frac{1}{a} \int_{(w-a)/2}^{(w+a)/2} \bar{f}_t(x) dx - \bar{f}_t(w/2) \right| \leq \alpha a^\gamma.$$

If $\bar{f}_t$ is not continuous at $w/2$, we rewrite (4) as

$$g(t) = \frac{1}{a} \int_{(w-a)/2}^{w/2} \bar{f}_t(x) dx + \frac{1}{a} \int_{w/2}^{(w+a)/2} \bar{f}_t(x) dx.$$

Using similar arguments, it is possible to show that the first integral converges to $(1/2)\bar{f}_t(w/2)$, whereas the second integral converges to $(1/2)\bar{f}_t((w/2)^+)$. Thus, when f is not continuous, the output of the filter defined by (4) converges to the output of the analog median filter defined by

$$g(t) := \frac{\bar{f}_t\left(\frac{w}{2}\right) + \bar{f}_t\left(\frac{w}{2}^+\right)}{2}. \quad (5)$$

Note that

$$\overline{f}_t\left(\frac{w^-}{2}\right) = \overline{f}_t\left(\frac{w}{2}\right)$$

because $\overline{f}$ is left continuous. In fact, the filter described by (5) can also be interpreted as follows: Define the “average sorting” of f as the average of its left-continuous and right-continuous sortings, and then redefine the median filter in terms of this “average sorting.”

A. Dependence on the Type of Sorting

Because the sorting is the basis for the definition of the median filter, different conventions regarding the sorting may in principle lead to median filters with distinct behaviors.

To fix ideas, consider median filters based on the left- and right-continuous sortings. Because the sortings may differ only at countably many points of discontinuity, we might be tempted to think that the responses of the two corresponding median filters to the same input signal f would differ only at countably many points (thus, they would be equal almost everywhere), but this is false: The outputs may differ not only at isolated points but throughout intervals. For example, it can be verified that the input signal described in [2] does in fact lead to such a situation, which can be properly understood in terms of the continuity properties of the sorting [3].

We give the simple example depicted in Fig. 9. Consider a running window of width $w = 2(b - a)$. Then, the sorting $\overline{f}_t$ of the windowed signal at t , f_t will be discontinuous at $w/2$ for $a \leq t \leq b$. Therefore, the outputs of the median filters based (for example) on the left-continuous and right-continuous sortings will differ from each other throughout that interval.

The following question remains: Is there a class of input signals for which the output of the median filter is independent of the underlying definition of sorting? Less strongly, is there a class of input signals for which the output of the median filter is, almost everywhere, independent of the definition of sorting? For brevity, these sets of signals will be denoted by A and B , respectively.

The answer to both questions is affirmative. The set of all continuous functions belongs to A because the sorting of a continuous function is unique and in the same Lipschitz class as the function itself, as shown in Appendix F.

However, the continuity constraint is unnecessarily strong. It is easy to construct examples of discontinuous functions with continuous sortings; the necessary and sufficient condition for the output of the median filter to be independent of the definition of the sorting is the continuity of $\overline{f}_t$ at $w/2$.

The class B strictly contains A . To check that this is true, consider a signal similar to the one depicted in Fig. 9 but of width $b - a \neq w/2$. The signal itself is, of course, discontinuous, but when $b - a < w/2$, the sorting of the windowed signal will be continuous at $w/2$ for all t . If $b - a > w/2$, there will be two exceptional points: $t = a$ and $t = b$. The sorting $\overline{f}_t$ will be continuous at $w/2$ for all other t .

Fig. 9. Sorting of the windowed version of f is discontinuous at one half the window width $w/2$ for all t satisfying $a \leq t \leq b$.

B. Relations Between the Digital and Analog Median Filters

The following results on the connection between the discrete-time and continuous-time median filters build on the sorting of step functions examined in Section III-C.

Let Δ be an arbitrary real number. Define

$$\chi_i(t) := \begin{cases} 1, & i\Delta < t \leq (i+1)\Delta \\ 0, & \text{otherwise.} \end{cases}$$

Any left-continuous piecewise constant function with possible discontinuities at $i\Delta$, $i \in \mathbb{Z}$ can now be expressed as

$$f_\Delta = \sum_i c_i \chi_i$$

for an adequate choice of the constants c_i .

1) *Odd-Length Case:* Consider an analog median filter with window length $w = (2N+1)\Delta$. Its response to the step function f_Δ at the time t will be denoted by $g_\Delta(t)$. The windowed signal at any $t_n := n\Delta + (\Delta/2)$, $n \in \mathbb{Z}$ will consist of exactly $2N+1$ steps of the step function f_Δ . On the other hand, the sorting of a step function can be obtained by sorting the steps by decreasing order (see Example 1 and Fig. 7). Hence, the sample $g_\Delta(t_n)$ can be obtained by sorting the $2N+1$ elements of the sequence c_i that fall inside the analog median window. The value of g_Δ at t_n is the median of these $2N+1$ values. Thus, it is equal to the n th output sample of a digital median filter with a window size of $2N+1$, which has the sequence c_i as an input.

Because the response g_Δ of the analog median filter to the step function f_Δ is also a step function, it is completely determined by the output samples of the digital median filter $g_\Delta(t_n)$, as explained above. In this sense, it is legitimate to mention “the step function determined by the output of the digital median filter.”

As discussed in Appendix E, if the sequence f_n converges monotonically, then the sortings $\overline{f}_n$ will also converge [see (14)]. This fact and the density of the set of step functions in several function spaces establishes the convergence of the outputs of the digital and analog median filter under suitable conditions. However, it is more useful to study the convergence with respect to the smoothness of the input signals. Therefore, we seek to obtain the order of approximation for functions in a given Lipschitz class.

Consider a continuous-time signal f with modulus of continuity ω_f . Let f_Δ be an approximating step function with coefficients c_i , which is obtained (for example) by sampling f . Then, as a consequence of the definition of the modulus of continuity

$$|f - f_\Delta| \leq \omega_f(\Delta).$$

The bound also applies to windowed versions of f and f_Δ , the sortings of which, when evaluated at $w/2$, are the response of the analog median filter to f and f_Δ . As shown in Appendix A, if $\alpha \leq f - g \leq \beta$, then $\alpha \leq \bar{f} - \bar{g} \leq \beta$ [see (7)]. Therefore, the responses g and g_Δ will also satisfy

$$|g - g_\Delta| \leq \omega_f(\Delta).$$

Because g_Δ can be obtained by applying a digital median filter of size $2N + 1$ to the sequence c_i and because the window size of the analog filter is $w = (2N + 1)\Delta$, we reach the following conclusions.

Proposition 1: Let the signal f be fed to an analog median filter with window length w . Let the samples of f , which are taken with sampling period Δ , be fed to a digital median filter of window size $2N + 1$. The output g of the analog median filter and the step function g_Δ reconstructed from the output of the digital median filter satisfy

$$|g - g_\Delta| \leq \omega_f\left(\frac{w}{2N + 1}\right)$$

where ω_f denotes the modulus of continuity of f . If f satisfies a Lipschitz condition of order γ with constant α

$$|g - g_\Delta| \leq \alpha \frac{w^\gamma}{(2N + 1)^\gamma}.$$

Thus, the step function determined by the output of the digital median filter converges uniformly to the output of the analog median filter, provided that $\Delta \rightarrow 0$ and $N \rightarrow \infty$, with $(2N + 1)\Delta = w$ (constant). $\square$

2) *Even-Length Case:* By construction, in the odd-length case, there are $2N + 1$ steps of width Δ inside the analog median window at the instants $t_n := n\Delta + (\Delta/2)$, $n \in \mathbb{Z}$. The sorting of the windowed step function at these time instants is always continuous at $w/2$ (indeed, it is constant in $(w/2 - \Delta/2, w/2 + \Delta/2)$).

The situation is only slightly different when the window length of the analog median is taken to be $2N\Delta$, in which case, the sorting will in general be discontinuous at $w/2$ (see Fig. 10). However, the size of the jump at the discontinuity is bounded by $\omega_f(\Delta)$, which converges to zero as $\Delta \rightarrow 0$. This leads to no essential changes in the previous conclusions.

V. EFFECTS OF ADDITIVE AND MULTIPLICATIVE NOISE

A. Noise Width

We now address the problem of determining the effect of noise on the analog median filter output. In order to do that, it is convenient to introduce the concept of signal width.

Definition 1: We say that the signal f has width δ if its sorting $\bar{f}$ vanishes for $x > \delta$. $\square$

It is possible to estimate the width of an impulsive signal. Assume, for example, that N points $\{t_i\}$ are randomly placed in an interval of length T . The probability of having k points in a subinterval of length α is

$$\binom{N}{k} p^k (1 - p)^{N-k}$$

Fig. 10. Sorting of a step function with n steps of width Δ is continuous or discontinuous at the middle point $w/2$, depending on n being odd or even.

where $p = \alpha/T$. If $N \gg 1$, $p \ll 1$, and k is close to Np , this can be approximated by

$$e^{-pN} \frac{(pN)^k}{k!}.$$

When N and T tend to infinity, with N/T constant and equal to d , this point process tends to a (ergodic) Poisson process with average d . The probability of having at most n points $\{t_i\}$ inside an interval of fixed length w is given by

$$P(n, d) = e^{-dw} \sum_{k=0}^n \frac{(dw)^k}{k!}.$$

Consider now a shot noise process that is generated by the superposition of functions h_k centered at the points t_k

$$s(t) := \sum_k h_k(t - t_k).$$

The h_k could as well be realizations of some stochastic process; we need only to assume that the measure of the support of the h_k is at most Δ . Consider now the windowed signal s_t obtained from s . If the window length is w , the probability that s_t has width less than or equal to $n\Delta$ is $P(n, d)$. This is plotted in Fig. 11.

In the median filter, as in other practical scenarios, the signals to be sorted have a maximum duration w (that is, they are time limited to w time units). Their sortings will necessarily vanish for $x > w$. We will concentrate on certain types of noise signals that will interact additively or multiplicatively with time-limited signals of width at most w , such as the windowed signals f_t on which the median filter is based. The relative values of δ and w suggest a distinction between small width or large width noise.

- Small width signals are signals with sortings that vanish for $x > \delta$ with $\delta \ll w$. Such signals are allowed to have arbitrary amplitudes, which are not necessarily small. This model is adequate to deal with additive, impulsive noise.
- Large width signals are signals f with sortings that satisfy $\bar{f}(x) = c$, where c is a real constant, throughout an interval of width $w - \delta$ with $\delta \ll w$. This model is adequate to deal with multiplicative noise. In that context, we may assume that $c = 1$ without loss of generality.

In addition to the classes of signals, we will also consider noise signals of large width, possibly up to w , but of instantaneous amplitude limited to $0 < \varepsilon \ll 1$.

Note that a windowed signal f_t can be of small width for all t , even if f itself has arbitrarily large width.

Note also that there is, in general, no relation between the width of a deterministic or stochastic signal and concepts such as energy or variance. It is sufficient to observe that the width

Fig. 11. Probability $P(n, d)$ that an impulsive signal multiplied by a rectangular window of length w has width less than or equal to $n\Delta$, plotted as a function of n , for several possible point densities d (in the range $0.1w$ to $10w$).

of a signal f is invariant under scaling, which does not change the support of $\bar{f}$.

B. Additive Noise

We wish to determine the output of the median filter when the input is a signal f to which a noise signal n has been added.

The response of the median filter to the signal-plus-noise $f + n$, which is evaluated at the time instant t , is $\overline{f_t + n_t}(w/2)$. The absolute value of the difference between this quantity and the response of the filter to the noiseless signal $g(t) = \bar{f}_t(w/2)$ is denoted by $E(t)$

$$E(t) := |\overline{f_t + n_t}(w/2) - \bar{f}_t(w/2)| = |\overline{f_t + n_t}(w/2) - g(t)|.$$

We intend to determine bounds for $E(t)$ in terms of the noise width and the smoothness of the input signals.

Let the windowed noise n_t be of small width δ . Under this constraint, f_t and $f_t + n_t$ differ on a set of measure at most δ . It is shown in Appendix B that the sortings of any functions f and g that differ on sets of measure at most δ satisfy $\bar{g}(x + \delta) \leq \bar{f}(x) \leq \bar{g}(x - \delta)$. Consequently

$$\bar{f}_t(w/2 + \delta) \leq \overline{f_t + n_t}(w/2) \leq \bar{f}_t(w/2 - \delta). \quad (6)$$

If $\bar{f}_t$ is constant in a sufficiently large neighborhood of $w/2$, the lower bound and the upper bound in (6) reduce to $g(t) = \bar{f}_t(w/2)$, and this leads to the following result.

Proposition 2: If $\bar{f}_t$ is constant throughout an interval I containing $[w/2 - \delta, w/2 + \delta]$, the output of the median filter at t will be unaffected by additive noise n_t of width bounded by δ : $E(t) = 0$. $\square$

Let f be a step function, with the steps more than w apart. Assume that there is one discontinuity at $t = 0$. We wish to examine the behavior of the median filter with window size w in the vicinity of this discontinuity in reference to the previous result.

Since the steps are separated by more than w , the sorting $\bar{f}_t$ is continuous at $w/2$ for each nonzero t satisfying $|t| < w/2$. In fact, $\bar{f}_t$ is constant throughout $((w/2) - \alpha, (w/2) + \alpha)$ for all t satisfying $\alpha < |t| < w/2$. By the previous result, the median

filter will reject any additive noise of width less than α , except possibly for values of t closer to the discontinuities than α .

If the instants where the discontinuities occur are known beforehand, so will the zones of uncertainty. The original step signal can then be recovered exactly by sampling the median filtered signal at the middle point of each step.

Subtracting from (6) the response of the filter to the noiseless signal $\bar{f}_t(w/2)$ leads to

$$E(t) \leq |\bar{f}_t(w/2 - \delta) - \bar{f}_t(w/2)|.$$

Therefore

$$E(t) \leq \omega_{\bar{f}_t}(\delta).$$

Since the sorting of a continuous function belongs to the same Lipschitz class as the function itself, as shown in Appendix F, we reach the following conclusion.

Proposition 3: If f satisfies a Lipschitz condition of order γ and constant α and the width of the noise n_t is bounded by δ , then

$$E(t) \leq \alpha \delta^\gamma. \quad \square$$

Assume now that the signal and the noise are non-negative. In Appendix D, we examine the sorting of the sum of two non-negative signals. In particular, applying (12), we conclude that

$$\overline{f_t + n_t}(w/2) \leq \bar{f}_t(w/2 - \delta) + \overline{n_t}(\delta).$$

Therefore

$$E(t) \leq \omega_{\bar{f}_t}(\delta) + \overline{n_t}(\delta)$$

and the invariance of the Lipschitz classes under sorting leads to the following conclusion.

Proposition 4: If $f \geq 0$ and the width of the noise n_t is bounded by δ , then

$$E(t) \leq \omega_f(\delta) + \overline{n_t}(\delta). \quad \square$$

If the width of the noise is below δ , $E(t) \leq \omega_f(\delta)$, as in the previous cases.

Assume now that the noise is the sum of a large width, low-amplitude component u , $|u(t)| \leq \varepsilon$, and a small width component v . In this case, we have

$$\overline{f_t + v_t}(w/2) - \varepsilon \leq \overline{f_t + u_t + v_t}(w/2) \leq \overline{f_t + v_t}(w/2) + \varepsilon.$$

Therefore

$$\bar{f}_t(w/2 + \delta) - \varepsilon \leq \overline{f_t + u_t + v_t}(w/2) \leq \bar{f}_t(w/2 + \delta) + \varepsilon.$$

Subtracting the response to f leads to

$$\begin{aligned} \bar{f}_t(w/2 + \delta) - \bar{f}_t(w/2) - \varepsilon &\leq \overline{f_t + u_t + v_t}(w/2) - \bar{f}_t(w/2) \\ &\leq \bar{f}_t(w/2 + \delta) - \bar{f}_t(w/2) + \varepsilon \end{aligned}$$

and we have the next proposition.

Proposition 5: Assume that the noise can be expressed as $n = u + v$, where $|u| \leq \varepsilon$, and v has width at most δ . Then

$$E(t) \leq \omega_f(\delta) + \varepsilon. \quad \square$$

C. Multiplicative Noise

In this section, we assume that the noise is multiplicative, that is, the noisy signal is fn , assuming that f is the noiseless signal. In this case

$$E(t) := |\overline{f_t n_t}(w/2) - \overline{f_t}(w/2)|.$$

Let the signal and the noise be non-negative. The sorting of the product of two functions is examined in Appendix C, where it is shown that

$$\overline{fg}(\alpha + \beta) \leq \overline{f}(\alpha)\overline{g}(\beta).$$

We conclude that

$$\overline{f_t n_t}(w/2) \leq \overline{f_t}(w/2 - \delta)\overline{n_t}(\delta)$$

and therefore

$$\begin{aligned} E(t) &\leq |\overline{f_t}(w/2 - \delta)\overline{n_t}(\delta) - \overline{f_t}(w/2)| \\ &\leq |\overline{f_t}(w/2 - \delta) - \overline{f_t}(w/2)|\overline{n_t}(\delta) + |\overline{n_t}(\delta) - 1|\overline{f_t}(w/2) \\ &\leq \omega_{\overline{f_t}}(\delta)\overline{n_t}(\delta) + \overline{f_t}(w/2)|\overline{n_t}(\delta) - 1|. \end{aligned}$$

Picking the smallest δ such that $\overline{n_t}(\delta) = 1$, and using once again the invariance of the Lipschitz classes, we see that

$$E(t) \leq \omega_f(\delta)$$

as for the case of additive noise. If $n_t(\delta)$ deviates from the unity by less than ε , then

$$E(t) \leq \omega_f(\delta)(1 + \varepsilon) + \varepsilon\overline{f_t}(w/2) = \omega_f(\delta)(1 + \varepsilon) + \varepsilon g(t).$$

VI. SOME OPEN ISSUES

The investigation that has been started may continue along several directions. The implementation of the analog median filters is one of these. To compute the output at a certain time instant t , it is necessary to find the smallest threshold y such that

$$\text{meas}\{x: f_t(x) > y\} = w/2.$$

The set S , where f_t exceeds y , can be determined with a comparison circuit, and its measure can be found by integrating the output of the comparator. The value determined could perhaps be used to adjust the comparison threshold to the comparator by means of a feedback loop. Optical implementations, which were asked for in [1], are also a possibility. The hybrid electronic/optical filter described in [5] was a step in this direction. Implementations similar to the one described in [12] may also be of interest: the circuit, which contains no sorting subsystem or clock signal, is based on nonlinear equations, the solution of which leads to the order statistics. The interesting sorting network reported in [4] is based on a nonlinear dynamical system,

and according to the authors, it is also able to provide the solution to the continuous-time rank filter.

On the theoretical side, and despite the usefulness of inequalities such as (9) or (12), the lack of subadditivity exhibited by the sorting can be overcome by replacing it with a related function, such as the Hardy–Littlewood function. It can essentially be obtained by integrating $\overline{f}$, as in

$$g(t) := \frac{1}{t} \int_0^t \overline{f}(x) dx$$

and might justify further research.

Very little was said regarding possible generalizations of the median filter to the continuous-time case, except for the α -trimmed or L filters, whose appearance is natural in this context. In fact, it is almost inevitable to consider other operators acting on the sorting, in addition to the pointwise evaluation that leads to the median filter. A large number of unexplored possibilities exist, involving, for example, hybrid linear/nonlinear filters.

There are also open issues regarding the behavior under noise of the analog median filter. The results given above depend on the apparently new concept of noise width, which naturally appears in this context. The possibility of studying the effects of impulsive or shot noise in the absence of knowledge regarding its amplitude statistics may justify a closer look at this concept.

APPENDIX

A. Bounding the Difference Between Two Sortings

If α and β are constants and $\alpha \leq f - g \leq \beta$, then

$$\boxed{\alpha \leq \overline{f} - \overline{g} \leq \beta.} \quad (7)$$

To see that (7) is true, note that $f \leq g$ implies $\overline{f} \leq \overline{g}$. Then, consider

$$g + \alpha \leq f \leq g + \beta.$$

Therefore

$$\overline{g + \alpha} \leq \overline{f} \leq \overline{g + \beta}.$$

It remains to be noted that

$$\overline{f + \alpha} = \overline{f} + \alpha$$

because

$$\overline{f + \alpha}(x) = \inf_y \{M_f(y - \alpha) < x\} = \inf_y \{M_f(y) < x\} + \alpha$$

[or $M_{f+\alpha}(y) = M_f(y - \alpha)$].

B. Functions Differing on Sets of Small Measure

If f and g differ on a set of measure at most δ , then

$$\boxed{\overline{g}(x + \delta) \leq \overline{f}(x) \leq \overline{g}(x - \delta)} \quad (8)$$

for $x > \delta$. Indeed, let $x_0 > \delta$ and $y_0 = \overline{f}(x_0)$. Denote by S the set of measure x_0 , where f exceeds y_0 . It is always possible to

find a set S' , where the function g also exceeds y_0 , possibly by removing from S a set of measure at most δ . The measure of S' will therefore satisfy

$$\text{meas}\{x: g(x) > y_0\} \geq x_0 - \delta$$

that is, $M_g(y_0) \geq x_0 - \delta$. Hence

$$y_0 \leq \bar{g}(x_0 - \delta)$$

and

$$\bar{f}(x_0) \leq \bar{g}(x_0 - \delta)$$

which is one of the inequalities in (8). To obtain the other, apply the same argument but reversing the role of the functions. This leads to

$$\bar{g}(x) \leq \bar{f}(x - \delta), \quad x > \delta$$

and (8) follows, after replacing x with $x + \delta$.

C. Distribution and Sorting of a Product

If f and g are non-negative, the inclusion

$$\{x: f(x)g(x) > \alpha\beta\} \subset \{x: f(x) > \alpha\} \cup \{x: g(x) > \beta\}$$

shows that

$$M_{fg}(\alpha\beta) \leq M_f(\alpha) + M_g(\beta).$$

Rewriting this in terms of the a and b defined by $\alpha = M_f(a)$ and $\beta = M_g(b)$ leads to

$$M_{fg}(ab) \leq M_f(a) + M_g(b) = \alpha + \beta.$$

Therefore

$$ab \geq \bar{f}\bar{g}(\alpha + \beta)$$

that is, $\bar{f}(\alpha)\bar{g}(\beta) \geq \bar{f}\bar{g}(\alpha + \beta)$, or

$$\boxed{\bar{f}\bar{g}(\alpha + \beta) \leq \bar{f}(\alpha)\bar{g}(\beta).} \quad (9)$$

D. Distribution and Sorting of a Sum

If f and g have disjoint supports, then $M_{f+g} = M_f + M_g$, but this is not generally true (see Fig. 6). The distribution is not even subadditive. The inequality $M_{f+g} \leq M_f + M_g$ fails to hold in general. However, the inclusion

$$\{x: f(x) + g(x) > y\} \subset \{x: f(x) > y/2\} \cup \{x: g(x) > y/2\}$$

holds true since $f(x_0) + g(x_0) > y$ for a certain x_0 implies that at least one of $f(x_0)$ and $g(x_0)$ exceeds $y/2$. This shows that

$$M_{f+g}(y) \leq M_f(y/2) + M_g(y/2). \quad (10)$$

To obtain an expression for the sorting of the sum, rewrite (10) in terms of $\alpha = M_f(y/2)$ and $\beta = M_g(y/2)$, that is

$$M_{f+g}(\bar{f}(\alpha) + \bar{g}(\beta)) \leq \alpha + \beta.$$

This suggests that

$$\overline{f+g}(\alpha + \beta) \leq \bar{f}(\alpha) + \bar{g}(\beta)$$

which is indeed true in the general case [that is, when α and β are not necessarily given by $\alpha = M_f(y/2)$ and $\beta = M_g(y/2)$, as assumed above]. The proof is based on

$$\overline{e^f}(x) = e^{\bar{f}(x)} \quad (11)$$

which follows from the more general result

$$\overline{\phi(f)}(x) = \phi(\bar{f}(x))$$

where ϕ is one-to-one. The distribution function of $\phi(f(x))$ is $M_f(\phi^{-1}(y))$. Denoting the composition of functions $f(g(x))$ by $f \circ g$, we see that the distribution function of $\phi \circ f$ is $M_f \circ \phi^{-1}$. Therefore

$$\overline{\phi \circ f} = (M_f \circ \phi)^{-1} = \phi \circ M_f^{-1} = \phi \circ \bar{f}$$

of which (11) is a special case. It turns out that

$$\begin{aligned} e^{\overline{f+g}(\alpha+\beta)} &= \overline{(e^f e^g)}(\alpha + \beta) \\ &= \overline{(e^f e^g)}(\alpha + \beta). \end{aligned}$$

Using (9)

$$\begin{aligned} \overline{(e^f e^g)}(\alpha + \beta) &\leq \overline{e^f}(\alpha) \overline{e^g}(\beta) \\ &\leq e^{\bar{f}(\alpha)} e^{\bar{g}(\beta)} \\ &= e^{\bar{f}(\alpha) + \bar{g}(\beta)}. \end{aligned}$$

We have established that

$$e^{\overline{f+g}(\alpha+\beta)} \leq e^{\bar{f}(\alpha) + \bar{g}(\beta)}$$

and hence

$$\boxed{\overline{f+g}(\alpha + \beta) \leq \bar{f}(\alpha) + \bar{g}(\beta)} \quad (12)$$

without any restrictions in α and β .

E. Convergence

If $f_n \rightarrow f$ monotonically ($c < f_n < f_{n+1}$) as $n \rightarrow \infty$, then $M_{f_n} \rightarrow M_f$ monotonically as $n \rightarrow \infty$. Indeed

$$\{x: f_n(x) > y\} \subset \{x: f(x) > y\}$$

and the union of the sets on the left-hand side is the set on the right-hand side. Therefore

$$M_{f_n}(y) \leq M_f(y), \quad M_{f_n}(y) \rightarrow M_f(y). \quad (13)$$

If $f_n \rightarrow f$ monotonically as $n \rightarrow \infty$, then $\bar{f}_n \rightarrow \bar{f}$ monotonically as $n \rightarrow \infty$. Indeed, from (13)

$$\bar{f}_n(x) \leq \bar{f}(x), \quad \bar{f}_n(x) \rightarrow \bar{f}(x). \quad (14)$$

F. Modulus of Continuity of f and $\bar{f}$

Let f satisfy a Lipschitz condition of order γ and constant α :

$$|f(a) - f(b)| \leq \alpha|a - b|^\gamma.$$

Then

$$M_f(y) - M_f(y + \varepsilon) \geq \left(\frac{\varepsilon}{\alpha}\right)^{1/\gamma} \quad (15)$$

provided that y has a neighborhood contained in the range of f . This condition ensures the existence of a $\delta > 0$ such that for all $\varepsilon < \delta$, the point $y + \varepsilon$ is in the range of f . To see that (15) holds, note that under the hypothesis, there will be at least one x_0 such that $y = f(x_0)$ [the horizontal line of height y will intersect the (continuous) graphic of f at least once]. When y increases to $y + \varepsilon$, the intersection will move to some x_1 with $y + \varepsilon = f(x_1)$. However

$$|f(x_0) - f(x_1)| = \varepsilon \leq \alpha|x_0 - x_1|^\gamma.$$

Therefore

$$|x_0 - x_1| \geq \left(\frac{\varepsilon}{\alpha}\right)^{1/\gamma}.$$

The measure of the set $S_y := \{x: f(x) > y\}$, when y increases by ε , decreases at least by the amount indicated on the right-hand side, and (15) follows.

A slightly more general result can be obtained using the modulus of continuity ω_f . It asserts that

$$M_f(y) - M_f(y + \varepsilon) \geq \omega_f^{-1}(\varepsilon).$$

To obtain a bound for the modulus of continuity of the sorting, rewrite (15) as

$$|M_f(a) - M_f(b)| \geq \left(\frac{|b - a|}{\alpha}\right)^{1/\gamma}$$

and set $x_0 = M_f(a)$ and $x_1 = M_f(b)$. Then

$$|a - b| \geq \left(\frac{\bar{f}(x_0) - \bar{f}(x_1)}{\alpha}\right)^{1/\gamma}$$

and therefore

$$\boxed{|\bar{f}(x_0) - \bar{f}(x_1)| \leq \alpha|x_0 - x_1|^\gamma.} \quad (16)$$

Thus, if f satisfies a certain Lipschitz condition, so will the sorting $\bar{f}$. This can be extended to the modulus of continuity: $\omega_{\bar{f}}(\delta) \leq \omega_f(\delta)$, which reduces to (16) in the Lipschitz case.

When f is discontinuous, M_f can be constant throughout intervals, and such bounds cannot be expected to exist.

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